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Optimal Investment in Advertising and Quality to Mitigate a Possible Product-Harm Crisis

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Abstract

Product-harm crises are the nightmare of any firm due to their disastrous effects on sales and image. These crises lead to loss of consumer trust, severe damage to brand reputation, extensive negative media coverage, legal and financial repercussions, decline in market share, negative impact on investor confidence, and increased regulatory scrutiny. Overall, product-harm crises pose significant challenges to companies, emphasizing the critical importance of effective risk management and crisis preparedness. The present paper proposes a new model to compute the optimal investment in quality and advertising in order to reduce the probability of occurrence of a possible product-harm crisis and mitigate its effects. This method uses stochastic control theory and can be used for both tangible products and services. An extension of this method is also proposed in order to take endogenously competition. This extension uses a game theoretical approach.

Keywords

Optimal Investment, Product-Harm Crisis, Stochastic Optimal Control, Game Theory, Advertising, Quality

1. Introduction

Many examples remind us that no company is immune from a product-harm crisis which can generate losses of several billion dollars. No sector is spared by this issue, we can mention, for example the car industry (Toyota), the food and beverage industry (Perrier) or the catering (Buffalo Grill). Between 2009 and 2010, Toyota had to carry out vehicle recalls due to quality-related issues. The crisis was triggered by reports of unintended acceleration in several Toyota models, which led to accidents, injuries, and even fatalities. In 1990, traces of benzene, a chemical compound potentially harmful to health, were detected in bottles of

Perrier mineral water. As a result, Perrier issued a voluntary recall of millions of bottles worldwide, and sales were temporarily suspended in many markets. In 2002, Buffalo Grill, a French restaurant chain, admitted of having imported British meat after the 1996 embargo related to mad cow disease. This caused an immediate drop in attendance of 40%. As evidenced by the case of Toyota, firms known to invest heavily in quality can be affected as well as less virtuous firms.

In the US, since many years, these crises are always accompanied by costly class action lawsuits which can be even more damaging than the decline in sales or the image degradation. Quite recently, the European Commission has promulgated a directive allowing class actions in Europe. To make things even worse, consumers now act as "consum-actors" and do not hesitate to organize boycotts. The case of Kitkat chocolate bars shows us that a powerful company may be forced to revise its production and communication under pressure from consumers. In the current climate, no company can ignore the impact of a product-harm crisis when making strategic decisions. Unfortunately, few tools are used to quantify the effects of a decision on a possible crisis. Currently, companies have a variety of tools to assess the impact of investments in normal circumstances but not the effects on a possible crisis. Today there is a real lack, and almost everything remains to be built.

Crisis management is an important area of management science and many articles and books have been written on this subject, for example (Bernstein, 2011), (Augustine, 1995) or (O'Donnell, 2009). Most of these publications offer precepts to follow when crises occur. Some also offer recommendations to prevent or mitigate a future crisis, without quantifying the effect of these recommendations. That said, for twenty years, researchers have proposed studies to quantify the consequences of a crisis. Some have used an experimental approach and have studied the effects that a crisis can have on consumer expectations (Dawar & Pillutla, 2000) or the brand loyalty (Stockmyer, 1996). Others have used empirical approaches to quantify the effects of a crisis on sales (see (Van Heerde et al., 2007) or (Cleeren et al., 2013)). However, all these studies analyse the crisis ex post and do not offer the manager a tool to measure the impacts of today's decisions on any future crisis. Other researchers have studied this problem and have proposed models allowing an ex ante analysis of crises. Using optimal control theory, they built models that calculate the optimal decision to make in an anticipatory manner while considering the effects of any future crisis. We can cite, among others, the pioneering work of (Rao, 1986) and later those of (Raman & Naik, 2004). Unfortunately, all these works assume that the crisis follows a Wiener process. This means that, for these models, the crisis is not sudden and violent but its outbreak is spread over time and its effects are the result of a multitude of small underlying crises. These models are obviously not realistic, but (Rubel et al., 2011) proposed a new model closer to reality. Adapting the work of (Boukas et al., 1990) and (Haurie & Moresino, 2006) in optimal control theory, they developed a model where crises are described by a Poisson process. The model proposed by Rubel et al. calculates the optimal investments to be made in advertising, while considering the effects of a possible crisis. More recently, Rubel (2018) proposed a version where competition is described with a game theoretical framework.

Dynamic advertising problems have been extensively studied since the pioneer work of (Nerlove & Arrow, 1962), (Vidale & Wolfe, 1957), (Kimball, 1957) or (Bass, 1969). A nice and complete literature review on the subject has been done by (Huang, et al., 2012). However, few works have been done to study dynamic advertising problems where the quality is a decision variable. (El Ouardighi & F. Pasin, 2006) seem to be the first to propose such a model. They were followed by (Nair & Narasimhan, 2006), (De Giovanni, 2011) and (El Ouardigh & Kogan, 2013, El Ouardighi et al., 2008).

The present paper proposes an extension of the model developed by Rubel et al. (2011). Indeed, we provide a new model to calculate the optimal investments in quality and advertising considering the effects of a possible product-harm crisis. We apply the numerical method proposed by (Kushner & Dupuis, 1992). This method relies on a discretization of time and space and allows to reformulate a stochastic control model into a Markov Decision Process (MDP). The solution of this MDP can be computed solving a linear program.

This paper is organized as follows. In the second section, we present the model. In the third section, we explain the numerical method used throughout this paper. The fourth section is dedicated to a numerical case study. The fifth section considers a model that takes endogenously the competition and finally, in the last section, further research directions are proposed.

2. The Model

We propose to extent the model proposed by Rubel et al. (2011), allowing investments in quality. Let $i=0$ denotes the precrisis regime and $i=1$ the postcrisis regime. Denote with S the sales and Q the quality. Following the extension of the Vidal-Wolf model proposed by (Sethi, 1983), the sales dynamics are given by

$$\frac{\mathrm{d}S}{\mathrm{d}t}=\beta_i\sqrt{Q(t)u_i\left(M(t)-S(t)\right)}-\delta_i S(t)-\varepsilon S(t)\left(1-Q(t)\right), \tag{1}$$

where M is the market size, u the investment in advertising, β the effectiveness, δ and ε decay rates. The quality dynamics are given by

$$\frac{\mathrm{d}Q}{\mathrm{d}t}=\alpha_i\sqrt{v_i\left(1-Q(t)\right)}-\mu_i Q(t), \tag{2}$$

where v denotes the investment in quality, α the effectiveness and μ the decay rate. The crisis follows a continuous time Markovian process with generator

$$q_{ij}=\xi_{i0}+\xi_{i1}Q,\quad i\neq j. \tag{3}$$

As usual, we denote with

$$q^i = \sum_{j \neq i} q_{ij} \equiv q(Q,i). \tag{4}$$

When a crisis occurs, the sales fall and the damage rate is denoted with Φ. In other words, the sale S falls to $(1-\Phi)S$, where of course $0 \leq \Phi \leq 1$. The profit function is given by

$$\pi(S,Q,u,v) = m_1 S - m_2 SQ - m_3 u - m_4 v, \tag{5}$$

where m_1 is the unit margin, m_2 the unit production price for quality, m_3 and m_4 investment costs. The objective is to maximize the discounted expected profits

$$V(S(0),Q(0),i) = \max_{u,v} E\left[\int_0^\infty e^{-\rho t} \pi(S,Q,u,v)\,dt\right] \tag{6}$$

with discount rate ρ.

For compactness we write $x = (S,Q)$, $w = (u,v)$, $\dot{x} = f(x,i,w)$, $\nabla V(x,i)$ the gradient of $V(x,i)$ with respect to x, $\Phi(x) = ((1-\Phi)S,Q)$ and $q(x,i) = q(Q,i)$. Applying standard dynamic programming analysis, we obtain the following Hamilton-Jacobi-Bellman (HJB) equations that provide sufficient conditions for the optimality (Fleming & Rishel, 1975):

$$\rho V(x,0) = \max_w \left\{\pi(x,w) + \nabla V(x,0)\cdot f(x,i,w) + \left[V(\Phi(x),1) - V(x,0)\right] q(x,0)\right\} \tag{7}$$

$$\rho V(x,1) = \max_w \left\{\pi(x,w) + \nabla V(x,1)\cdot f(x,i,w) + \left[V(x,0) - V(x,1)\right] q(x,1)\right\} \tag{8}$$

3. Numerical Method

In general, the HJB system of Equations (7-8) cannot be solved analytically. However, using Kushner and Dupuis method (Kushner & Dupuis 1992), a numerical approximation can be computed. The space of the variable x_k is discretized with mesh h_k. In other words, the variable x_k belongs to the grid $\mathcal{G}_k = \left\{x_k^{\min}, x_k^{\min} + h_k, x_k^{\min} + 2h_k, \cdots, x_k^{\max}\right\}$. Let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$, we denote with $\partial\mathcal{G}$ the grid's boundary. We approximate the partial derivatives by finite differences taken in the direction of the flow:

$$\frac{\partial V(x,i)}{\partial x_k} \to \begin{cases} \dfrac{V(x+e_k h_k,i) - V^j(x,i)}{h_k} & \text{if } f_k(x,i,w) \geq 0 \\ \dfrac{V(x,i) - V(x-e_k h_k,i)}{h_k} & \text{if } f_k(x,i,w) < 0, \end{cases} \tag{9}$$

where e_k the unit vector of the k-th axis. Let

$$f_k^+(x,i,w) = \max\left\{0, f_k(x,i,w)\right\} \tag{10}$$

$$f_k^-(x,i,w) = \max\left\{0, -f_k(x,i,w)\right\}. \tag{11}$$

For $x_1 \in \mathcal{G}_1$, we define $\hat{x}_1 = \min\left\{y \in \mathcal{G} \mid y \geq x_1(1-\Phi)\right\}$ and $\breve{x}_1 = \max\left\{y \in \mathcal{G} \mid y \leq x_1(1-\Phi) \text{ or } y = x_1^{\min}\right\}$. For $\hat{x}_1 \neq \breve{x}_1$, we approximate $V(\Phi(x),1)$ as follows

$$V\left(\Phi(x),1\right) \rightarrow \frac{x_1-\breve{x}_1}{\hat{x}_1-\breve{x}_1}V\left(\hat{x},1\right)+\frac{\hat{x}_1-x_1}{\hat{x}_1-\breve{x}_1}V\left(\breve{x},1\right). \tag{12}$$

Substituting the differences to the partial derivatives in Equations (7-8), one obtains

$$\begin{aligned}\rho V\left(x,0\right)=\max_w\Bigg\{&\pi\left(x,w\right)+\left[\frac{x_1-\breve{x}_1}{\hat{x}_1-\breve{x}_1}V\left(\hat{x},1\right)+\frac{\hat{x}_1-x_1}{\hat{x}_1-\breve{x}_1}V\left(\breve{x},1\right)-V\left(x,0\right)\right]q\left(x,0\right)\\&+\sum_k\left(\frac{V\left(x+e_kh_k,0\right)-V\left(x,0\right)}{h_k}f_k^+\left(x,0,w\right)\right.\\&\left.+\frac{V\left(x,0\right)-V\left(x-e_kh_k,0\right)}{h_k}f_k^-\left(x,0,w\right)\right)\Bigg\}.\end{aligned} \tag{13}$$

and

$$\begin{aligned}\rho V\left(x,1\right)=\max_w\Bigg\{&\pi\left(x,w\right)+\left[V\left(x,0\right)-V\left(x,1\right)\right]q\left(x,1\right)\\&+\sum_k\left(\frac{V\left(x+e_kh_k,1\right)-V\left(x,1\right)}{h_k}f_k^+\left(x,1,w\right)\right.\\&\left.+\frac{V^1\left(x\right)-V^1\left(x-e_kh_k\right)}{h_k}f_k^-\left(x,1,w\right)\right)\Bigg\}.\end{aligned} \tag{14}$$

Let

$$\omega=\max_{x,w,i}q\left(x,i\right)+\sum_k\frac{\left|f_k\left(x,i,u\right)\right|}{h_k} \tag{15}$$

Define the interpolation interval

$$\Delta=\frac{1}{\rho+\omega} \tag{16}$$

and the discounting factor

$$r=\frac{\omega}{\rho+\omega}. \tag{17}$$

We define transition probabilities $\Pi\left(x,y,i,j,w\right)$ as follows:

• When $x\in\mathcal{G}\setminus\partial\mathcal{G}$ the transition probabilities from *x* to any neighbouring value $x\pm e_kh_k$, if we stay in state *i*, are given by

$$\Pi\left(x,x\pm e_kh_k,i,i,w\right)=\frac{f_k^{\pm}\left(x,i,w\right)}{\omega h_k}. \tag{18}$$

• When $x\in\mathcal{G}\setminus\partial\mathcal{G}$ the transition probabilities to stay in the same state are given by

$$\Pi\left(x,x,i,i,u\right)=1-\frac{q\left(x,i\right)+\sum_k\frac{\left|f_k\left(x,i,u\right)\right|}{h_k}}{\omega}. \tag{19}$$

• If $\hat{x}_1=\breve{x}_1$, the transition probabilities from $i=0$ to $i=1$, are given by

$$\Pi(x,\hat{x},0,1,u)=\frac{q(x,0)}{\omega}. \tag{20}$$

If $\hat{x}_1 \neq \breve{x}_1$, we have

$$\Pi(x,\hat{x},0,1,u)=\frac{x_1-\breve{x}_1}{\hat{x}_1-\breve{x}_1}\frac{q(x,0)}{\omega} \tag{21}$$

$$\Pi(x,\breve{x},0,1,u)=\frac{\hat{x}_1-x_1}{\hat{x}_1-\breve{x}_1}\frac{q(x,0)}{\omega} \tag{22}$$

• The transition probabilities for a jump from $i=1$ to $i=0$ are given by

$$\Pi(x,x,1,0,u)=\frac{q(x,1)}{\omega}. \tag{23}$$

• On the boundary $\partial\mathcal{G}$ of the grid, the probabilities are defined according to a reflecting boundary scheme.

• All the other transition probabilities are zero.

The possible transitions are represented in **Figure 1**, **Figure 2**. Substituting this in Equations (13-14) and rearranging terms leads to the following dynamic programming (DP) equations:

$$V(x,i)=\max_{w}\left\{\pi(x,w)\Delta+r\sum_{y,j}\Pi(x,y,i,j,w)V(y,j)\right\}. \tag{24}$$

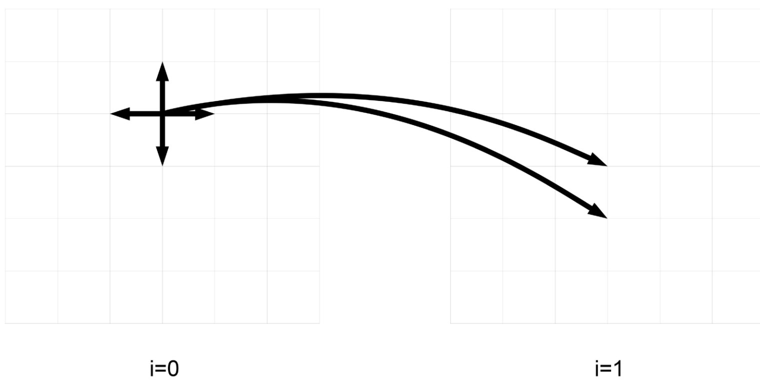

Figure 1. Transitions in the grid set (precrisis regime).

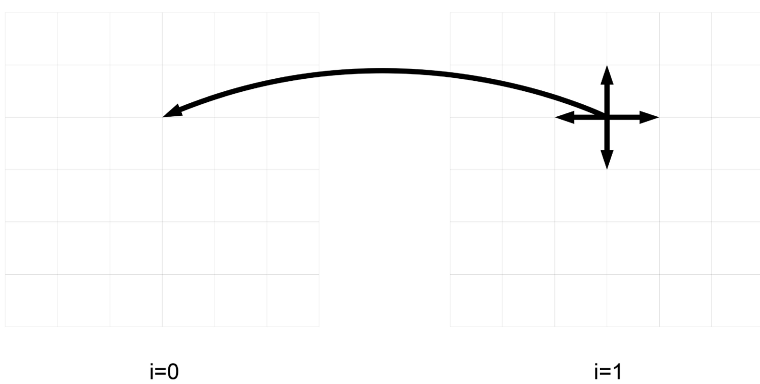

Figure 2. Transitions in the grid set (postcrisis regime).

If ones assume that the control belongs to a finite set, these equations are the DP equation of a discrete state MDP. It is well established that the approximating MDPs lead to approximations of the value functions that converge weakly toward the continuous time solutions of the DP equations. Using the classical verification theorems of DP, we conclude that the optimal solutions obtained from the approximating MDPs provide a ε-optimal solution to the continuous time problem.

To solve these DP equations, we can use a linear programming formulation (see Puterman (2005)). Let $B(x,i) \geq 0$ with $\sum_{x,i} B(x,i) = 1$. We have to solve the following problem:

$$\max \sum_{x,i,w} \pi(x,w) \Delta Z(x,i,w) \tag{25}$$

s.t.

$$\sum_{w} Z(y,j,w) - r \sum_{x,i,w} \Pi(x,y,i,j,w) Z(x,i,w) = B(y,j), \quad y \in \mathcal{G},\ j = 0,1 \tag{26}$$

$$Z(x,i,w) \geq 0, \tag{27}$$

Then the optimal policy is given by

$$D(w \mid x,i) = \frac{Z(x,i,w)}{\sum_{v} Z(x,i,v)}. \tag{28}$$

$D(w \mid x,i) = 1$ if w is the optimal decision for the state (x,i) and zero otherwise.

4. Numerical Experiment

The main objective of this numerical experiment is to provide the proof of concept for our method. Using fictive but realistic data, we also illustrate the kind of results the method can provide. We run the model for the set of data given in **Table 1** and the grid given in **Table 2**.

Table 1. Data.

	Precrisis regime	Postcrisis regime		
α	0.5	0.5	ρ	0.06
μ	0.1	0.1	m_1	100
β	0.05	0.05	m_2	0.5
δ	0.1	0.3	m_3	20
ε	0.01	0.03	m_4	1
ξ_0	0.5	2	Φ	0
ξ_1	−0.005	0.05		
M	100	100		

Table 2. Grid.

	Minimum	Maximum	Mesh
S	0	100	4
Q	0	100	4
u	0	100	10
v	0	100	10

Figure 3 and Figure 4 show the optimal policy for both regimes. The graphic on the left of Figure 3 shows for the precrisis regime, the optimal investment in advertising given the state of sales and quality. As expected, we notice that the higher the quality, the higher the advertising and the lower the sales the higher the advertising. The graphic on the right of Figure 3 shows the optimal investment in quality given the state of sales and quality. As expected, we see that the lower the quality, the higher the investment in quality and the lower the sales the higher the investment in quality. Figure 4 shows similar results for the postcrisis regime. The graphic on the left of Figure 5 shows the evolution in time of the sales for two initial values. The graphic on the left of Figure 5 shows the evolution in time of the quality for two initial values. This figure shows that the trajectories are attracted by the so-called turnpike. This figure shows two trajectories in the precrisis regime, one starting from the point S(0) = 50 and Q(0) = 10, the other starting from the point S(0) = 90 and Q(0) = 80. We see distinctly that both converge to the turnpike S = 76.2 and Q = 47.3. Similarly, Figure 6 shows the turnpike for the postcrisis regime (S = 60.1 and Q = 69.5). As expected, sales decline in crisis times. Interestingly we see that, in case of crisis, it is optimal to invest more in quality in order to quit as soon as possible this turmoil phase. Finally, Figure 7 shows the steady state probabilities for both regimes.

5. Competition

In this section, we consider a model that takes endogenously competition. More precisely, we consider a market where two firms compete for the same customers. The model is identical to the model described in Section 2 except for two differences. First, Equation (1) is enriched as follows

$$\frac{\mathrm{d}S_l}{\mathrm{d}t} = \beta_{il}\sqrt{Q_l(t)u_i\left(M(t)-S_l(t)-S_{-l}(t)\right)} - \delta_{il}S_l(t) - \varepsilon S_l(t)\left(1-Q_l(t)\right). \quad (29)$$

In this notation, the subscript l denotes firm $l \in \{1,2\}$ and the subscript $-l$ denotes the other firm. This Equation links the sales dynamics of both firms. As we are in a non cooperative competition we have to find a Nash equilibrium. Consequently, the objective function (Equation (6)) writes now

$$W_l\left(w_l, w_{-l}, i\right) = E\left[\int_0^\infty \mathrm{e}^{-\rho t}\pi\left(S_l, Q_l, u_l, v_l\right)\mathrm{d}t\right] \quad (30)$$

and

$$W_l\left(w_l^*, w_{-l}^*, i\right) \geq W_l\left(w_l, w_{-l}^*, i\right), \quad l \in \{1,2\}. \tag{31}$$

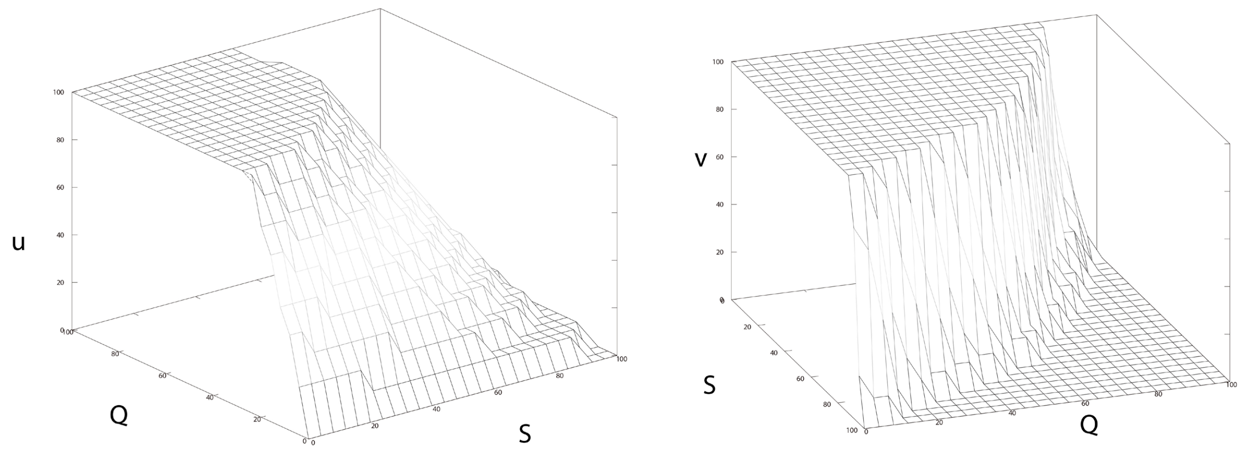

Figure 3. Optimal investment in advertising (left) and quality (right) for the precrisis regime.

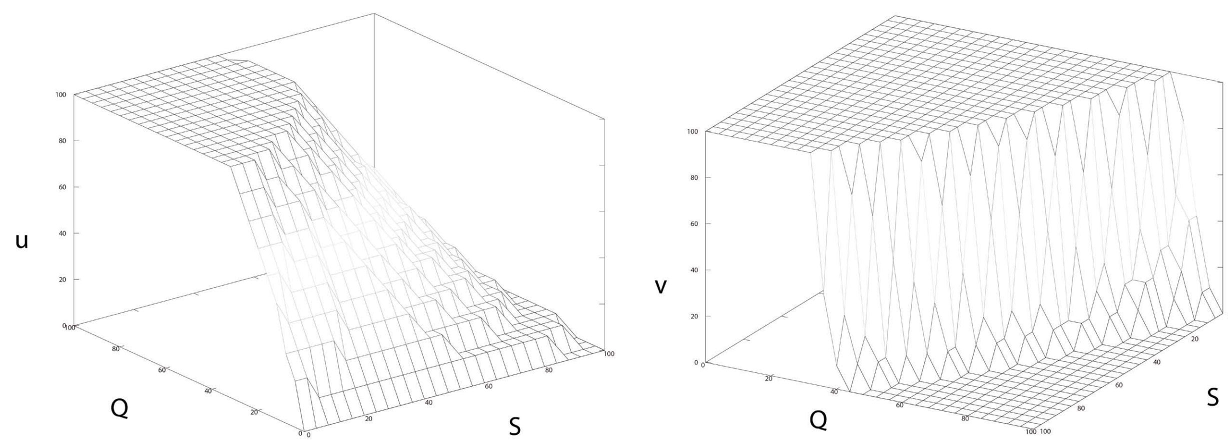

Figure 4. Optimal investment in advertising (left) and quality (right) for the postcrisis regime.

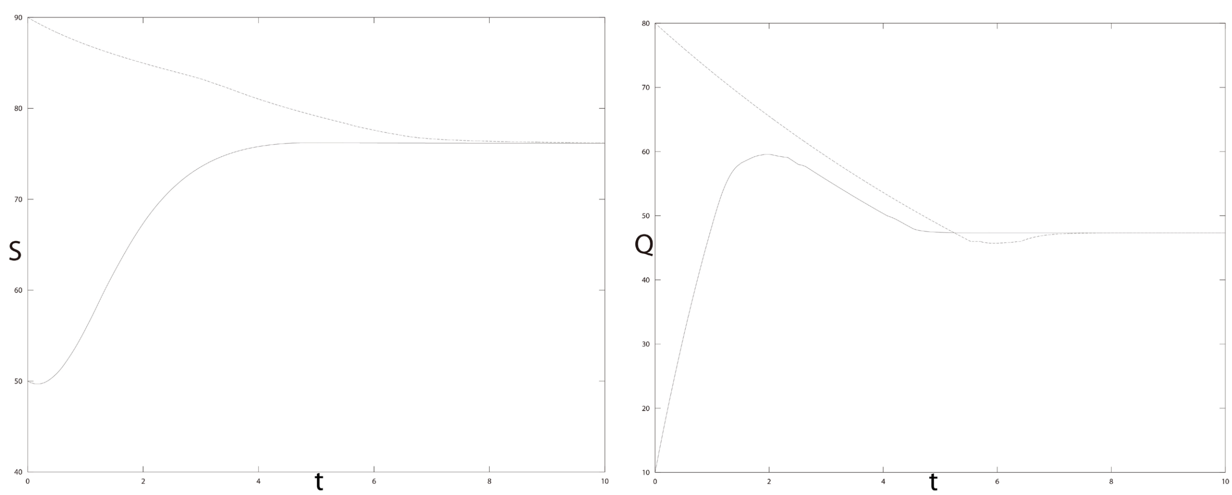

Figure 5. Two optimal trajectories with different initial values (precrisis regime). Left: sales; right: quality.

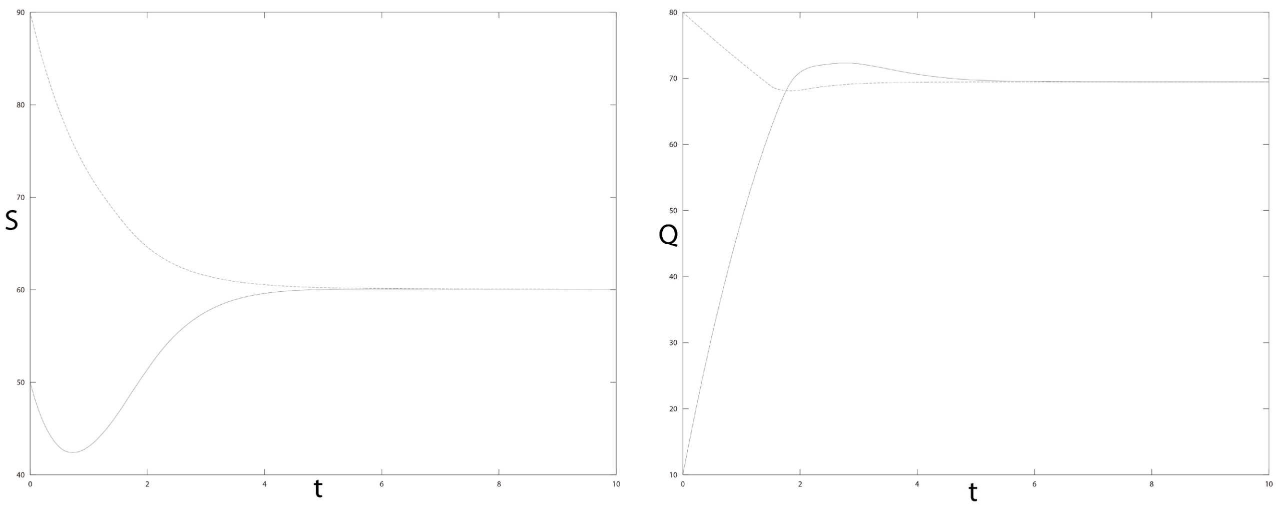

Figure 6. Two optimal trajectories with different initial values (postcrisis regime). Left: sales; right: quality.

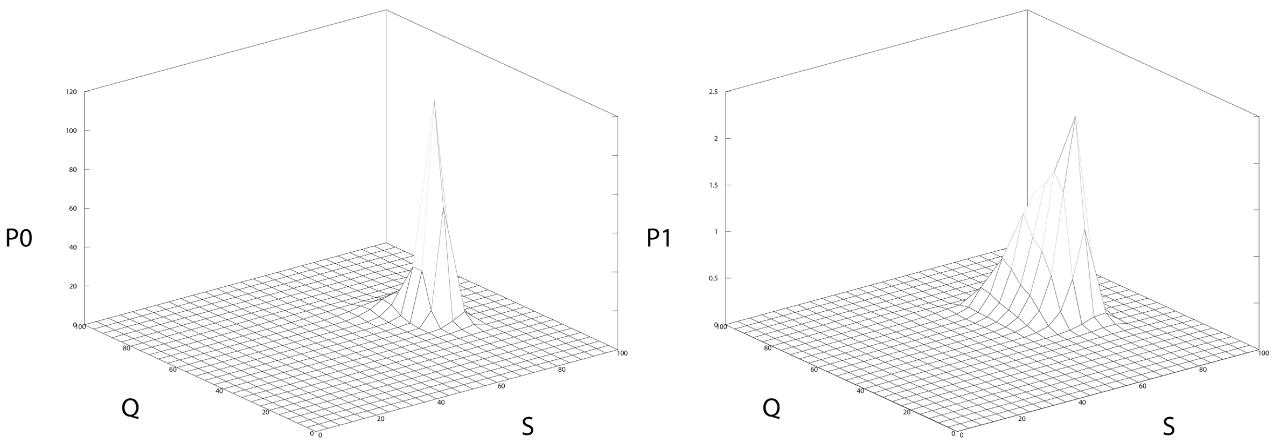

Figure 7. Steady state probabilities. Left: precrisis regime; right: postcrisis regime.

It is illusory to solve this model without numerical method. Applying the same method as in Section 3, we would have to solve a non zero sum competitive Markov decision Process. The solution could be found through the equilibrium of a bimatrix game. Unfortunately, this problem is equivalent to a mixed-integer program and we face the famous curse of dimensionality. Knowing that the trajectories are attracted by the turnpikes, to get around this problem, we consider the following approximation. When there is a change of regime, we suppose that the competing firm's trajectory jump instantaneously to the corresponding turnpike. This approximation is valid if the speed to reach the turnpikes is fast compared to the average time spend in a regime.

Then, concretely our model reduces to a fixed point problem where the operators are given by two MDP similar to the one described in Section 3. Although the properties of the operators are not known, our numerical experiment suggests they are contracting. This permits us to use a standard iterative method to find the fixed points.

Table 3. The turnpikes for Firm 1.

Firm 1	Firm 2	
	precrisis	postcrisis
precrisis	$\bar{S} = 46.4$ $\bar{Q} = 34.0$	$\bar{S} = 53.8$ $\bar{Q} = 38.8$
postcrisis	$\bar{S} = 36.0$ $\bar{Q} = 65.9$	$\bar{S} = 42.6$ $\bar{Q} = 67.2$

For the numerical illustration, we used a symmetrical model where the parameters are the same as those of Section 4. For the results, we display only the turnpikes values, as the controls and the trajectories have the similar behaviour as those in Section 4. **Table 3** shows the turnpike for Firm 1. As the model is symmetrical, the turnpikes form Firm 2 is identical. These results show us that when a firm is in a postcrisis regime, it is optimal to increase the quality. This confirms what we observed in Section 4. In addition to this information, this game theory model suggests that each firm will take advantage when the other firm has a crisis. Indeed, we see distinctly that the quality is increased when the other firm has problems.

6. Conclusion

In this paper, we have proposed an original method to compute the optimal investment in quality and advertising in order to reduce the probability of occurrence of a possible product-harm crisis and mitigate its effects. This method is an extension of the method proposed by Rubel et al. and uses a stochastic control theory approach. A numerical approximation of the optimal policy is computed using the method proposed by Kushner and Dupuis.

We also proposed an extension of our model in order to consider the competition endogenously. More precisely we proposed a game theory model where two firms compete for the same market. As this latter model faced the famous curse of dimensionality, we used a trick: both firms are linked only with their turnpikes' states.

For our fictive case, numerical results show that in case of crisis, it is optimal to increase the investments in quality in order to quit as soon as possible this turmoil phase. For the competing case, the model shows that each firm will take advantage when the other firm has a crisis by increasing its investment efforts.

Although this method enhances existing methods, it could be improved on two points. First, our method, like other numerical methods, suffers from the curse of dimensionality. Indeed, for cases with many different products, the model could be numerically intractable. To get around this problem it could be judicious to confine our attention on the turnpikes and develop a new method for approximating the optimal policy. Another possibility would be to discretise the time but not the space and use, for instance, a least squares solution method (Kek et al., 2017). Second, for the extension where competition is considered, we assumed that only the turnpikes link the different actors. It would be interesting to enrich the approach and consider a link at any points of time.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

Augustine, N. R. (1995). Managing the Crisis You Tried to Prevent. *Harvard Business Review, 73,* 147-158.

Bass, F. M. (1969). A New Product Growth for Model Consumer Durables. *Management Science, 15,* 215-227. https://doi.org/10.1287/mnsc.15.5.215

Bernstein, J. (2011). *Manager's Guide to Crisis Management.* McGraw-Hill.

Boukas, E. K., Haurie, A., & Michel, P. (1990). An Optimal Control Problem with a Random Stopping Time. *Journal of Optimization Theory and Applications, 64,* 471-480. https://doi.org/10.1007/BF00939419

Cleeren, K., van Heerde, H. J., & Dekimpe, M. G. (2013). Rising from the Ashes: How Brands and Categories Can Overcome Product-Harm Crisis. *Journal of Marketing, 77,* 58-77. https://doi.org/10.1509/jm.10.0414

Dawar, N., & Pillutla, M. M. (2000). Impact of Product-Harm Crises on Brand Equity: The Moderating Role of Consumer Expectations. *Journal of Marketing Research, 37,* 215-226. https://doi.org/10.1509/jmkr.37.2.215.18729

De Giovanni, P. (2011). Quality Improvement vs. Advertising Support: Which Strategy Works Better for a Manufacturer? *European Journal of Operational Research, 208,* 119-130. https://doi.org/10.1016/j.ejor.2010.08.003

El Ouardigh, F., & Kogan, K. (2013). Dynamic Conformance and Design Quality in a Supply Chain: An Assessment of Contracts' Coordinating Power. *Annals of Operations Research, 211,* 137-166. https://doi.org/10.1007/s10479-013-1414-4

El Ouardighi, F., & Pasin, F. (2006). Quality Improvement and Goodwill Accumulation in a Dynamic Duopoly. *European Journal of Operational Research, 175,* 1021-1032. https://doi.org/10.1016/j.ejor.2005.06.020

El Ouardighi, F., Jorgensen, S., & Pasin, F. (2008). A Dynamic Game of Operations and Marketing Management in a Supply Chain. *International Game Theory Review, 10,* 373-397. https://doi.org/10.1142/S021919890800200X

Fleming, W. H., & Rishel, R. W. (1975). *Deterministic and Stochastic Control.* Springer. https://doi.org/10.1007/978-1-4612-6380-7

Haurie, A., & Moresino, F. (2006). A Stochastic Control Model of Economic Growth with Environmental Disaster Prevention. *Automatica, 42,* 1417-1428. https://doi.org/10.1016/j.automatica.2005.10.018

Huang, J., Leng, M. M., & Liang, L. P. (2012). Recent Developments in Dynamic Advertising Research. *European Journal of Operational Research, 220,* 591-609. https://doi.org/10.1016/j.ejor.2012.02.031

Kek, S. L., Li, J., & Teo, K. L. (2017). Least Squares Solution for Discrete Time Nonlinear

Stochastic Optimal Control Problem with Model-Reality Differences. *Applied Mathematics, 8,* 1-14. https://doi.org/10.4236/am.2017.81001

Kimball, G. E. (1957). Some Industrial Applications of Military Operations Research Methods. *Operations Research, 5,* 201-204. https://doi.org/10.1287/opre.5.2.201

Kushner, H. J., & Dupuis, P. G. (1992). *Numerical Methods for Stochastic Control Problems in Continuous Time.* Springer. https://doi.org/10.1007/978-1-4684-0441-8

Nair, A., & Narasimhan, R. (2006). Dynamics of Competing with Quality- and Advertisingbased Goodwill. *European Journal of Operational Research, 175,* 462-474. https://doi.org/10.1016/j.ejor.2005.05.015

Nerlove, M., & Arrow, K. J. (1962). Optimal Advertising Policy under Dynamic Conditions. *Economica, 29,* 129-142. https://doi.org/10.2307/2551549

O'Donnell, K. (2009). In a Crisis, Action Matters. *Marketing News 22.*

Puterman, M. L. (2005). *Markov Decision Processes: Discrete Stochastic Dynamic Programming.* John Wiley & Sons.

Raman, K., & Naik, P. A. (2004). Long-Term Profit Impact of Integrated Marketing Communications Program. *Review of Marketing Science, 2,* Article 8. https://doi.org/10.2202/1546-5616.1014

Rao, R. C. (1986). Estimating Continuous Time Advertising-Sales Models. *Marketing Science, 5,* 125-142. https://doi.org/10.1287/mksc.5.2.125

Rubel, O. (2018). Profiting from Product-Harm Crises in Competitive Markets. *European Journal of Operational Research, 265,* 219-227. https://www.sciencedirect.com/science/article/pii/S0377221717306598 https://doi.org/10.1016/j.ejor.2017.07.032

Rubel, O., Naik, P. A., & Srinivasan, S. (2011). Optimal Advertising When Envisioning a Product-Harm Crisis. *Marketing Science, 30,* 1048-1065. https://doi.org/10.1287/mksc.1110.0679

Sethi, S. P. (1983). Deterministic and Stochastic Optimization of a Dynamic Advertising Model. *Optimal Control Applications and Methods, 4,* 179-184. https://doi.org/10.1002/oca.4660040207

Stockmyer, J. (1996). Brands in Crisis: Consumer Help for Deserving Victims. *Advances in Consumer Research, 23,* 429-435.

Van Heerde, H. J., Helsen, K., & Dekimpe, M. G. (2007). The Impact of a Product-Harm Crisis on Marketing Effectiveness. *Marketing Science, 26,* 230-245. https://doi.org/10.1287/mksc.1060.0227

Vidale, M. L., & Wolfe, H. B. (1957). An Operations-Research Study of Sales Response to Advertising. *Operations Research, 5,* 370-381. https://doi.org/10.1287/opre.5.3.370